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GRADED SUBALGEBRAS OF AFFINE KAC–MOODY ALGEBRAS*

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ABSTRACT

We determine the maximal graded subalgebras of affine Kac–Moody algebras. We also show that the maximal graded subalgebras of loop algebras are essentially loop algebras.

1. Introduction

The maximal subalgebras of the simple finite-dimensional Lie algebras over $\mathbb{C}$ (the complex numbers) were determined in classical works of Dynkin [D1],[D2]. Since the late sixties there has been considerable interest in some classes of infinite dimensional simple Lie algebras, most notably affine Kac–Moody algebras, and various classical results were extended to the infinite-dimensional case (see [K]). However, while the representation theory of affine Kac–Moody algebras is now fairly developed, it seems that virtually nothing is known on the subalgebra

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structure of these objects (or on the subgroup structure of Kac–Moody groups). In this paper we initiate the study of the maximal subalgebras of affine Kac–Moody algebras and of related objects, such as loop algebras. Our results are fairly general, in that they hold over arbitrary fields F, though they take a particularly simple form in the important case $F = \mathbb{C}$. The case of finite fields is also of some interest: it can be applied in the context of Hausdorff dimension for pro-p groups (see [BSh]).

Given a finite-dimensional Lie algebra $\mathcal{G}$ over F, consider the infinite-dimensional F-algebra

$$L(\mathcal{G}) = \mathcal{G} \otimes_F F[t, 1/t],$$

where $F[t, 1/t]$ is the ring of Laurent polynomials. Then (derived) affine Kac–Moody algebras (corresponding to an indecomposable extended Cartan matrix) $\tilde{L}$ can be realized as central extensions of $L(\mathcal{G})$ by a 1-dimensional center Z, where $\mathcal{G}$ is a simple finite-dimensional Lie algebra over F. The study of the maximal subalgebras of $\widetilde{L}$ is therefore reduced to the study of the maximal subalgebras of $L(\mathcal{G})$. The latter algebra is $\mathbb{Z}$-graded, and so it is natural to focus first on its maximal graded subalgebras. Note that, since $L(\mathcal{G})$ is finitely generated, it follows from Zorn's Lemma that every proper graded subalgebra of $L(\mathcal{G})$ can be extended to a maximal one. Two obvious maximal graded subalgebras of $L = L(\mathcal{G})$, defined by

$$L^+ = \mathcal{G} \otimes F[t] \quad \text{and} \quad L^- = \mathcal{G} \otimes F[1/t],$$

will play some role below. If M is a graded subalgebra of $L(\mathcal{G})$, then we may write $M = \bigoplus_{n\in\mathbb{Z}} M_n \otimes t^n$, where M_n are linear subspaces of $\mathcal{G}$. We say that M is **periodic** if there exists $k > 0$ such that $M_n = M_{n+k}$ for all integers $n \in \mathbb{Z}$. We say that M is **weakly periodic** if there exists $k > 0$ such that $M_n = M_{n+k}$ for all but finitely many integers $n \in \mathbb{Z}$. The minimal such k is called the **period** of M. Subalgebras M and N of L are said to be **commensurable** if $M \cap N$ has finite codimension in M and in N.

We can now state our first result.

THEOREM 1.1: *Let $\mathcal{G}$ be a central simple finite-dimensional Lie algebra over a field F. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n \otimes t^n$ be a maximal graded F-subalgebra of $L(\mathcal{G})$. Then M is weakly periodic. Moreover, either M is commensurable with L^+ or with L^-, or M is periodic.*

Recall that a simple Lie algebra $\mathcal{G}$ over F is said to be central simple if F coincides with the centroid $\mathrm{Cent}(\mathcal{G})$ of $\mathcal{G}$, which consists of all elements $T \in$

$\mathrm{End}_F\,\mathcal{G}$ satisfying $[T(x),y] = T([x,y])$ $(x,y \in \mathcal{G})$. The assumption that $\mathcal{G}$ is central simple is essential, as shown below. However, this assumption can be replaced by the weaker condition, that every element in the centroid of $\mathcal{G}$ has some power in F. For example, this is the case if F is finite. Thus, if F is finite, or algebraically closed, then it suffices to require that $\mathcal{G}$ is simple.

Theorem 1.1 is used to provide an explicit description of the maximal graded subalgebras of $L(\mathcal{G})$. We need the following standard

Definition: Let k be a positive integer, and let $\mathcal{G} = \bigoplus_{i=0}^{k-1} \mathcal{G}_i$ be a $\mathbb{Z}_k$-grading of $\mathcal{G}$ (where $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$). Let α be the k-tuple $(\mathcal{G}_0, \ldots, \mathcal{G}_{k-1})$. Define

$$L(\mathcal{G},k,\alpha) = \bigoplus_{n\in\mathbb{Z}} \mathcal{G}_{n \bmod k} \otimes t^n.$$

Then $L(\mathcal{G},k,\alpha)$ is said to be a **loop algebra** on $\mathcal{G}$ (corresponding to α). Central extensions of loop algebras are often regarded as twisted affine Kac–Moody algebras.

If F contains a primitive kth root of unity ζ_k, we shall also identify α with the semisimple automorphism (of order dividing k) acting on $\mathcal{G}_i$ as multiplication by ζ_k^i $(i = 0,\ldots,k-1)$.

Conversely, if $\zeta_k \in F$ and $\alpha \in \mathrm{Aut}\,\mathcal{G}$ is a semisimple automorphism satisfying $\alpha^k = 1$, then α defines a $\mathbb{Z}_k$-grading of $\mathcal{G}$ (which we also denote by α), and we have $L(\mathcal{G},k,\alpha) = \bigoplus_{n\in\mathbb{Z}} M_n \otimes t^n$, where

$$M_n = \{x \in \mathcal{G}\colon \alpha(x) = \zeta_k^n x\}.$$

We allow the trivial $\mathbb{Z}_k$-grading $\alpha = (\mathcal{G},0,\ldots,0)$ (corresponding to the automorphism $\alpha = 1$), which gives rise to the loop algebra $L(\mathcal{G},k,\alpha) = \mathcal{G}\otimes F[t^k, 1/t^k]$.

THEOREM 1.2: *Let F be any field and let $\mathcal{G}$ be a simple finite-dimensional Lie algebra over F. Let M be a maximal graded F-subalgebra of $L(\mathcal{G})$. Denote the centroid of $\mathcal{G}$ by K. Then one of the following holds:*

(i) *M is commensurable with L^+ or with L^-.*

(ii) a. *$M = \mathcal{H} \otimes F[t,1/t]$, where $\mathcal{H}$ is a maximal subalgebra of $\mathcal{G}$.*

(ii) b. *For some $\lambda \in K^*$ and a maximal subalgebra $\mathcal{H}$ of $\mathcal{G}$ satisfying $K\cdot\mathcal{H} = \mathcal{G}$ we have*

$$M = \oplus_{n\in\mathbb{Z}} \lambda^n \mathcal{H} \otimes t^n.$$

(iii) *$M = L(\mathcal{G},p,\alpha)$ for some prime p and a $\mathbb{Z}_p$-grading α of $\mathcal{G}$.*

Remarks: (1) Conversely, the subalgebras of types (ii)a, (ii)b and (iii) (as well as L^+ and L^-) are all maximal.

(2) Case (ii)b can be regarded as a twisted version of (ii)a. It does not occur if $\mathcal{G}$ is central simple over F.

(3) Maximal subalgebras of type (ii)b are usually not periodic. As an explicit example, take $F = \mathbb{Q}$ (the rational numbers), $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{G} = \mathrm{sl}_2(K)$ (regarded as an F-algebra), $\mathcal{H} = \mathrm{sl}_2(F)$, and $\lambda = \sqrt{2} + 1$.

(4) It follows from Theorem 1.2 that, if $\mathcal{G}$ is central simple, then every maximal graded subalgebra M of $L(\mathcal{G})$ has prime period or period 1.

(5) Suppose $F = \mathbb{C}$. Then, according to the theorem, in order to determine all the possibilities for M (up to commensurability), one needs to know

(a) the maximal subalgebras of $\mathcal{G}$, and

(b) the elements of prime order in $\mathrm{Aut}\,\mathcal{G}$.

This data is available in the literature (see Dynkin [D1, D2] and Kac [K, Chapter 8]), and so Theorem 1.2 is rather satisfactory.

It is interesting that, while our main results are Lie theoretic, the method of proof involves associative algebras (the main tool being central polynomials for matrix rings). This method is flexible enough to allow various extensions and variations. For example, we shall also determine the maximal graded subalgebras of loop algebras in general, as well as the maximal graded subalgebras of infinite codimension in $\mathbb{N}$-graded algebras such as $\mathcal{G} \otimes tF[t]$. The description of these subalgebras has applications in the study of Hausdorff dimension in groups such as $\mathrm{SL}_d(F_p[[t]])$ [BSh].

This paper is organized as follows. In Section 2 we prove Theorem 1.1, using central polynomials for matrix algebras as our main tool. Theorem 1.2 is proved in Section 3. In Section 4 we consider various extensions and variations. This is where the results for loop algebras and $\mathbb{N}$-graded algebras are presented. This section also contains some explicit examples which arise in group-theoretic contexts.

2. Central polynomials and periodicity

Let $\mathcal{G}$ be a finite-dimensional simple Lie algebra over a field F. It is well known that the centroid K of $\mathcal{G}$ is a field which is a finite extension of F, and that $\mathcal{G}$ is central simple over K. This reduces, to some extent, the study of simple algebras to that of central simple algebras.

LEMMA 2.1: *Let $\mathcal{G}$ be a finite-dimensional central simple Lie algebra over the field F. Set $d = \dim \mathcal{G}$. Then $\mathrm{ad}\,\mathcal{G}$ generates $\mathrm{End}_F\,\mathcal{G} \cong M_d(F)$ as an associative* *algebra.*

Proof: Let R be the associative algebra generated by $\operatorname{ad}\mathcal{G}$ in $\operatorname{End}_F \mathcal{G}$, which we identify with $M_d(F)$. Then R has a faithful simple module, namely $\mathcal{G}$, and so R is a primitive ring. Note that $\operatorname{End}_R(\mathcal{G}) = F$ (which we identify with the scalar matrices λI in $M_d(F)$), since $\mathcal{G}$ is central simple. By Jacobson's density theorem (see, for instance, [P, p. 220]) it follows that $R = M_d(F)$. ∎

Recall that, by a theorem of Razmyslov [R] (see also Formanek [F]), $R = M_d(F)$ has a multilinear central polynomial, namely, there exists a multilinear polynomial $q(x_1, \ldots, x_m)$ such that the values of q in R lie in the center $Z(R) = F$, and q does not vanish on R. Note that we can assume that the coefficients of q lie in the prime subfield F_0 of F. Indeed, if q_0 is a central polynomial for $M_d(F_0)$, then it is also a central polynomial for $M_d(F)$.

Let $b_1, \ldots, b_d$ be a basis for $\operatorname{ad}\mathcal{G}$. Then R is spanned over F by monomials of the form $b_{i_1} \cdots b_{i_k}$ $(k \geq 0)$. This implies that, for some choice of monomials P_j $(j = 1, \ldots, m)$ of the above form, we have

$$q(P_1, \ldots, P_m) \neq 0.$$

In other words, letting

$$f(y_{11}, \ldots, y_{1k_1}, \ldots, y_{m1}, \ldots, y_{mk_m}) = q(y_{11} \cdots y_{1k_1}, \ldots, y_{m1} \cdots y_{mk_m}),$$

we obtain a multilinear polynomial f in y_{ij} over the prime subfield of F, whose values on $\operatorname{ad}\mathcal{G}$ are scalars and are not all zero.

Note that $m = \deg q$ is bounded in terms of d (in fact we have $m = 2d^2 - 1$ in Razmyslov's construction). It is clear that we can choose the monomials P_j in such a way that their degrees are bounded in terms of d. Moreover, we can require that a particular basis element, say b_1, occurs in each of the monomials P_j. This is because $M_d(F) = M_d(F)b_1M_d(F)$, so $M_d(F)$ is spanned by monomials involving b_1.

We introduce the following

Definition: An (associative, non-commutative) polynomial $f(y_1, \ldots, y_N)$ will be called a **central polynomial** for $\mathcal{G}$ if

$$0 \neq f(\operatorname{ad}\mathcal{G}, \ldots, \operatorname{ad}\mathcal{G}) \subseteq \operatorname{Cent}(\mathcal{G}) = F.$$

We have proved the following:

PROPOSITION 2.2: *Let $\mathcal{G}$ be a central simple Lie algebra over a field F and let $d = \dim \mathcal{G}$. Then*

(i) *$\mathcal{G}$ has a multilinear central polynomial $f(y_1, \ldots, y_N)$ defined over the prime subfield of F, such that $N = \deg f$ is d-bounded.*

(ii) *In particular, if $b_1, \ldots, b_d$ is a basis for $\operatorname{ad} \mathcal{G}$, then there are indices $i_1, \ldots, i_N \in \{1, \ldots, d\}$ and a non-zero element $\lambda \in F$ such that*

$$f(b_{i_1}, \ldots, b_{i_N}) = \lambda I.$$

(iii) *In part* (ii) *above we can require that $i_1 = 1$.*

We need the following easy result whose proof is sketched below.

LEMMA 2.3: *Let $\mathcal{G}$ be a finite-dimensional simple Lie algebra over F, and let $L = L(\mathcal{G})$. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n \otimes t^n$ be a maximal subalgebra of L. Then*

(i) *M is infinite-dimensional.*

(ii) *If $M_n = \mathcal{G}$ (resp. $M_{-n} = \mathcal{G}$) for all large $n > 0$, then M is commensurable with L^+ (resp. L^-).*

Proof: For part (i), suppose M is finite-dimensional, and choose a positive integer k such that $M_n = 0$ for all integers n whose absolute value is $\geq k/2$. Then M can easily be extended to a periodic subalgebra K with period k such that $M < K < L$, and this violates the maximality of M.

For part (ii) fix a non-zero element $x \in \mathcal{G}$. Then the series of subspaces $x \cdot \operatorname{ad}(\mathcal{G})^k$ ($k \geq 1$) forms an ascending chain (as $\mathcal{G}$ is perfect), so it must stabilize at $k = \dim \mathcal{G}$. Since $\mathcal{G}$ is simple, we conclude that $x \cdot \operatorname{ad}(\mathcal{G})^d = \mathcal{G}$, where $d = \dim \mathcal{G}$. Now suppose $M_n = \mathcal{G}$ for all $n \geq N$, and choose n such that $M_n \neq 0$, Then, considering Lie products of type $[M_n, M_{n_1}, \ldots, M_{n_d}]$ where $n_i \geq N$, and using the equality $M_n \operatorname{ad}(\mathcal{G})^d = \mathcal{G}$, we see that $M_l = \mathcal{G}$ for all $l \geq n + dN$. Therefore, if $M_n \neq 0$ for infinitely many negative integers n, then $M_n = \mathcal{G}$ for all n, so $M = L$, which is impossible. It follows that $M_n = 0$ for almost all $n < 0$, and M is commensurable with L^+. ∎

Proof of Theorem 1.1: Let $F, \mathcal{G}, L, M$ be as in the formulation of the theorem.

CASE 1: $\sum_{n\in\mathbb{Z}} M_n \neq \mathcal{G}$. It is clear that the sum on the left hand side is a (proper) subalgebra of $\mathcal{G}$, and so it is contained in a maximal subalgebra of $\mathcal{G}$, say $\mathcal{H}$. Then $M \subseteq \mathcal{H} \otimes F[t, 1/t]$, and since M is maximal we have equality. In particular, M is periodic of period 1.

CASE 2: $\sum_{n\in\mathbb{Z}} M_n = \mathcal{G}$. Then we can choose elements $a_1 \in M_{k_1}, \ldots, a_d \in M_{k_d}$ which form a basis for $\mathcal{G}$ (where $k_1, \ldots, k_d$ are suitable integers). Setting $b_i = \operatorname{ad} a_i$ we obtain a basis $b_1, \ldots, b_d$ for $\operatorname{ad}\mathcal{G}$.

The idea now is to apply Proposition 2.2. Let f be as in the conclusion of the proposition. Then $f(b_{i_1}, \ldots, b_{i_N}) = \lambda I$ for some $\lambda \in F^*$. Let $m_i = a_i \otimes t^{k_i} \in M$. Since f is multilinear we have (in $\operatorname{End}_F M$)

$$f(\operatorname{ad} m_{i_1}, \ldots, \operatorname{ad} m_{i_N}) = \lambda I \otimes t^{k_{i_1}+\cdots+k_{i_N}}. \tag{1}$$

Setting $k = k_{i_1} + \cdots + k_{i_N}$ we conclude that, for every n,

$$M_{n+k} \otimes t^{n+k} \supseteq (M_n \otimes t^n) f(\operatorname{ad} m_{i_1}, \ldots, \operatorname{ad} m_{i_N}) = \lambda M_n \otimes t^{n+k}.$$

Since $\lambda M_n = M_n$ we conclude that

$$M_{n+k} \supseteq M_n \quad \text{for all } n. \tag{2}$$

The inclusion (2) can be used to prove the periodicity assertion, provided k is non-zero. To achieve this we distinguish between some cases.

CASE 2.1: *$M_n \neq 0$ for infinitely many $n > 0$.* We claim that we can choose the basis a_i in such a way that k_1 is much larger than d and $k_2, \ldots, k_d$. Indeed, starting with a basis $a_1, \ldots, a_d$ and corresponding degrees $k_1, \ldots, k_d$ as above, choose an arbitrarily large integer n such that $M_n \neq 0$. If $a \in M_n$ is a non-zero element, then another basis for $\mathcal{G}$ can be formed by replacing a suitable element a_i by a. Reordering the new basis if necessary we obtain the claim.

Now, assuming $i_1 = 1$ and that N is d-bounded as we may, we can therefore arrange that $k = k_1 + k_{i_2} + \cdots + k_{i_N}$ is positive (and is in fact as large as we like).

Next, given an integer l, consider the series of subspaces M_{l+ki} ($i \in \mathbb{Z}$). It is an ascending chain (by (2)), so it stabilizes. Let K_l denote the limit (which equals the union of all the subspaces M_{l+ki}). Let $K = \bigoplus_{l\in\mathbb{Z}} K_l \otimes t^l$. It is clear that K is periodic with period dividing k, and that $M \subseteq K$. If $K = L$ then we must have $M_n = \mathcal{G}$ for all large $n > 0$, and this implies that M is commensurable with L^+ by Lemma 2.3.

So suppose $K \neq L$. Then $M = K$ by maximality, so M is periodic.

CASE 2.2: *$M_{-n} \neq 0$ for infinitely many $n > 0$.* This case is resolved in exactly the same manner.

CASE 2.3: *$M_n = 0$ for almost all integers n.* In this case M is finite-dimensional, and this contradicts part (i) of Lemma 2.3.

The proof of Theorem 1.1 is complete. ∎

Remark: It is easy to see that the theorem remains valid under the weaker assumption that, for any λ in the centroid K of $\mathcal{G}$, there exists $l > 0$ such that $\lambda^l \in F$. Indeed, consider $\mathcal{G}$ as a central simple Lie algebra over K, and note that the central polynomial obtained in 2.2 is already defined over F. This enables us to apply the above arguments and to conclude that $M_{n+k} \supseteq \lambda M_n$ for some $\lambda \in K^*$ and for all n. While M_n may not be closed under multiplication by λ, we have

$$M_{n+kl} \supseteq \lambda^l M_n = M_n,$$

assuming $\lambda^l \in F$. Thus (2) holds with kl instead of k, and the rest of the proof goes through.

Example 2.4: Let F be any field of characteristic $\neq 2$ and let $\mathcal{G} = \mathrm{sl}_2(F)$. Let e, f, h be the usual basis for $\mathcal{G}$. Let $L = \mathcal{G} \otimes_F F[t, 1/t]$. For each positive integer l define a graded subalgebra $M = \bigoplus_{n\in\mathbb{Z}} M_n \otimes t^n$ as follows:

$$M_n = \mathcal{G} \quad \text{if } n \geq l, \quad M_n = \langle e, h\rangle \quad \text{if } 0 \leq n \leq l-1,$$
$$M_n = \langle e\rangle \quad \text{if } -l \leq n \leq -1, \quad \text{and} \quad M_n = 0 \quad \text{if } n < -l.$$

It can be verified that M is a maximal graded subalgebra of L. Letting l vary we obtain infinitely many maximal graded subalgebras of L which are all commensurable with L^+.

3. Determination of the maximal graded subalgebras

This section is devoted to the proof of Theorem 1.2. We shall first consider the easier case, when $\mathcal{G}$ is central simple.

LEMMA 3.1: *Let $\mathcal{G}$ be a central simple Lie algebra over F, and let $M \subset L(\mathcal{G})$ be a maximal graded subalgebra. Suppose M is periodic of period $l > 1$. Then $M = L(\mathcal{G}, l, \alpha)$ for some α (namely, M is a loop algebra on $\mathcal{G}$).*

Proof: Note that

$$\sum_{n=0}^{l-1} M_n = \mathcal{G}, \tag{3}$$

for otherwise M is contained in a subalgebra of type $\mathcal{H} \otimes F[t, 1/t]$ and is not maximal (since $l > 1$). It remains to show that (3) defines a $\mathbb{Z}_l$-grading of $\mathcal{G}$, namely, that the sum $\sum_{i=0}^{l-1} M_i$ is direct.

To prove this, let $f(y_1, \ldots, y_N)$ be a multilinear central polynomial for $\mathcal{G}$. Let $a_i \in M_{k_i}, b_i = \operatorname{ad} a_i, m_i = a_i \otimes t^{k_i} \in M$ be as in the proof of Theorem 1.1. We may choose $0 \leq k_i \leq l-1$ since M has period l. As in the proof of 1.1, we may also assume that $k = k_{i_1} + \cdots + k_{i_N} > 0$ and $f(b_{i_1}, \ldots, b_{i_N}) \neq 0$. This implies $M_n = M_{n+k}$ for all large n (hence for all n, since M is periodic). It follows that l divides k.

Now suppose by contradiction that the sum (3) is not direct. Without loss of generality, suppose

$$M_0 \cap (M_1 + \cdots + M_{l-1}) \neq 0. \tag{4}$$

Then we can choose the basis a_i so that $a_1 \in M_0$ (namely $k_1 = 0$). Assuming $i_1 = 1$ as we may, we obtain $f(b_1, b_{i_2}, \ldots, b_{i_N}) \neq 0$. By (4) we can express a_1 as a linear combination of elements $c_j \in M_j$ $(j = 1, \ldots, l-1)$, and since f is multilinear we have $f(\operatorname{ad} c_j, b_{i_2}, \ldots, b_{i_N}) \neq 0$ for some j. Replacing $m_1 = a_1 \otimes t^{k_1} = a_1 \otimes 1$ by $m_1' = c_j \otimes t^j \in M$ in (1), we obtain

$$f(\operatorname{ad} m_1', \operatorname{ad} m_{i_2}, \ldots, \operatorname{ad} m_{i_N}) = \lambda' I \otimes t^{k+j},$$

where $\lambda' \in F^*$. This yields $M_n = M_{n+k+j}$ for all n. Therefore l divides $k+j$. Since we also have $l|k$ we conclude that $l|j$, which is impossible (as $1 \leq j \leq l-1$). It follows that $\mathcal{G} = M_0 \oplus \cdots \oplus M_{l-1}$, so the lemma is proved. ∎

We need some notation. Let $\alpha = (\mathcal{G}_0, \ldots, \mathcal{G}_{k-1})$ be a $\mathbb{Z}_k$-grading of $\mathcal{G}$. Let l be a divisor of k, and let $m = k/l$. Denote by α^m the $\mathbb{Z}_l$-grading

$$\begin{aligned}(\mathcal{G}_0 + \mathcal{G}_l + \cdots + \mathcal{G}_{(m-1)l}, \mathcal{G}_1 + \mathcal{G}_{l+1} + \cdots + \mathcal{G}_{(m-1)l+1}, \cdots,\\ \mathcal{G}_{l-1} + \mathcal{G}_{2l-1} + \ldots + \mathcal{G}_{(m-1)l+l-1}).\end{aligned}$$

Note that if α is interpreted as an automorphism, then its mth power indeed corresponds to the above grading.

The following result is straightforward.

Lemma 3.2: *$L(\mathcal{G}, k, \alpha) \subseteq L(\mathcal{G}, l, \beta)$ if and only if l divides k and $\beta = \alpha^{k/l}$.*

Lemma 3.3: *Let $M = \bigoplus_{n \in \mathbb{Z}} M_n \otimes t^n$ be a graded subalgebra of $L(\mathcal{G})$, where $\mathcal{G}$ is central simple. Suppose M is a loop algebra on $\mathcal{G}$. Then M is maximal in $L(\mathcal{G})$ if and only if it has prime period.*

Proof: Write $M = L(\mathcal{G}, k, \alpha)$. Suppose first that k is not prime. Let $1 < l < k$ be a divisor of k. Then

$$L(\mathcal{G}) \supset L(\mathcal{G}, l, \alpha^{k/l}) \supset L(\mathcal{G}, k, \alpha) = M,$$

and so M is not maximal.

Conversely, suppose k is prime. Let K be a maximal graded subalgebra containing M. Then K cannot be commensurable with L^+ (otherwise we will have $M_{-n} = 0$ for all large n, and so $M = 0$, a contradiction). Similarly, K is not commensurable with L^-. Applying Theorem 1.1 we conclude that K is periodic. Its period must be greater than 1, otherwise $K = \mathcal{H} \otimes F[t, 1/t] \supseteq M$, which is impossible. Applying Lemma 3.1 we conclude that K is a loop algebra on $\mathcal{G}$, say $K = L(\mathcal{G}, l, \beta)$ for some $l > 1$. Applying Lemma 3.2 we see that l divides k and that $\beta = \alpha^{k/l}$. Since k is prime we must have $l = k$, and so $\beta = \alpha$ and $M = K$ is maximal. The result follows. ∎

Proof of Theorem 1.2: Let $M \subset L(\mathcal{G})$ be a maximal graded subalgebra. If $\mathcal{G}$ is central simple over F, then it follows from the above lemmas that M satisfies one of conditions (i), (ii)a, (iii) in the conclusion of the theorem.

Suppose now that $\mathcal{G}$ is not central simple and let K be the centroid of $\mathcal{G}$. Note that the action of K on $\mathcal{G}$ can be extended naturally to an action of K on $L(\mathcal{G})$. Consider the subspace $\widetilde{M} = K \cdot M$ which is spanned by the elements $T(x)$ $(T \in K, x \in M)$. Then $\widetilde{M}$ is a subalgebra of $L(\mathcal{G})$ satisfying $M \subseteq \widetilde{M} \subseteq L$, and so either $M = \widetilde{M}$ or $\widetilde{M} = L$. In the first case we can consider $\mathcal{G}$ as a central simple K-algebra and M as a maximal graded K-subalgebra, thus reducing to the central simple case, which was already treated. It therefore remains to consider the case $\widetilde{M} = L$.

In this case we have $K \cdot M_n = \mathcal{G}$ for all n. In particular, M_0 contains a basis $a_1, \ldots, a_d$ for $\mathcal{G}$ over K. Applying Proposition 2.2 for $\mathcal{G}$ as a central simple K-algebra, we obtain

$$f(\operatorname{ad} a_{i_1}, \ldots, \operatorname{ad} a_{i_N}) \neq 0,$$

where f is a multilinear central polynomial for $\mathcal{G}$. Now, since M_1 also contains a basis for $\mathcal{G}$ over K, we can find $a \in M_1$ such that

$$f(\operatorname{ad} a, \operatorname{ad} a_{i_2}, \ldots, \operatorname{ad} a_{i_N}) \neq 0.$$

Hence there exists $\lambda \in K^*$ such that

$$f(\operatorname{ad}(a \otimes t), \operatorname{ad} a_{i_2}, \ldots, \operatorname{ad} a_{i_N}) = \lambda I \otimes t.$$

Since $a \otimes t, a_{i_2}, \ldots, a_{i_N} \in M$ and f is defined over F, it follows that

$$M_{n+1} \supseteq \lambda M_n \quad \text{for all } n \in \mathbb{Z}. \tag{5}$$

Set $K_n = \lambda^{-n} M_n$ ($n \in \mathbb{Z}$). Then it follows from (5) that K_n is an ascending chain of subspaces of $\mathcal{G}$. Let $\mathcal{H} = \bigcup_{n\in\mathbb{Z}} K_n$. It follows from the definition of K_n that $\mathcal{H}$ is a subalgebra of $\mathcal{G}$, and since $\{K_n\}$ form a chain we have $K_n = \mathcal{H}$ for all sufficiently large n. Note that $\mathcal{H} \neq \mathcal{G}$. Indeed, assuming $\mathcal{H} = \mathcal{G}$ we would have $M_n = \mathcal{G}$ for all large n, so M is commensurable with L^+ by 2.3, and therefore $\widetilde{M} = K \cdot M \neq L$, a contradiction.

Since $M_n = \lambda^n K_n \subseteq \lambda^n \mathcal{H}$ for all n, we have $M \subseteq \sum_{n\in\mathbb{Z}} \lambda^n \mathcal{H} \otimes t^n$. The right hand side is a proper subalgebra of L, so the maximality of M yields

$$M = \sum_{n\in\mathbb{Z}} \lambda^n \mathcal{H} \otimes t^n.$$

It also follows from the maximality of M that $\mathcal{H}$ is a maximal subalgebra of $\mathcal{G}$. Finally, we have $K \cdot \mathcal{H} = K \cdot M_0 = \mathcal{G}$. Thus M is as in part (ii)b.

Theorem 1.2 is proved. ∎

4. Loop algebras and $\mathbb{N}$-graded algebras

Algebras of type $L(\mathcal{G})$ can be regarded as loop algebras of period 1. However, it turns out that our method provides a description of the maximal graded subalgebras of any loop algebra. For simplicity we restrict our attention to the case where the underlying finite-dimensional algebra $\mathcal{G}$ is central simple over F. Let $L = L(\mathcal{G}, k, \alpha)$ be a loop algebra associated with $\mathcal{G}$ with respect to the $\mathbb{Z}_k$-grading $\mathcal{G} = \bigoplus_{i=0}^{k-1} \mathcal{G}_i$. Then we can define subalgebras L^+ and L^- by

$$L^+ = L \cap (\mathcal{G} \otimes F[t]), \quad L^- = L \cap (\mathcal{G} \otimes F[1/t]).$$

Furthermore, with each graded subalgebra $\mathcal{H} = \bigoplus_{i=0}^{k-1} \mathcal{H}_i$ of $\mathcal{G}$ (where $\mathcal{H}_i \subseteq \mathcal{G}_i$) we can associate the algebra $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{n \bmod k} \otimes t^n$, which we denote (by a slight abuse of notation) by $L(\mathcal{H}, k, \alpha)$. Since $\mathcal{H}$ need not be simple, the algebra $L(\mathcal{H}, k, \alpha)$ may not be a loop algebra in the strict sense of the word, though it can be regarded as a 'generalized' loop algebra.

We can now state

THEOREM 4.1: *Let $\mathcal{G}$ be a central simple finite-dimensional Lie algebra over a field F. Let M be a maximal graded subalgebra of the loop algebra $L = L(\mathcal{G}, k, \alpha)$. Then one of the following holds:*

(i) *M is commensurable with L^+ or with L^-.*
(ii) *$M = L(\mathcal{H}, k, \alpha)$, where $\mathcal{H}$ is a maximal graded subalgebra of $\mathcal{G}$.*
(iii) *$M = L(\mathcal{G}, pk, \beta)$ for some prime p and a $\mathbb{Z}_{pk}$-grading β of $\mathcal{G}$ such that $\beta^p = \alpha$.*

We omit the proof of this result, since it follows our previous arguments almost verbatim, and no new ideas are involved. Note that Theorem 4.1 shows that all maximal graded subalgebras of $L(\mathcal{G},k,\alpha)$ are weakly periodic; in fact, excluding those of type (i), they are all periodic, and can be regarded as (generalized) loop algebras.

In some group-theoretic contexts it is important to look at the strictly positive part of loop algebras, and to investigate their graded subalgebras. In this case the algebras in question are $\mathbb{N}$-graded, and their maximal graded subalgebras have finite codimension and are not of great interest. The real interest lies in the graded subalgebras of infinite codimension which are maximal with respect to these properties. Note that it follows from Zorn's Lemma that every graded subalgebra of infinite codimension is contained in a maximal one (this is because graded subalgebras of finite codimension in the positive part of a loop algebra are finitely generated). A description of these subalgebras can be obtained using our methods.

THEOREM 4.2: *Let $L = L(\mathcal{G},k,\alpha)$ be as in Theorem 4.1. Let M be a graded subalgebra of $L\cap(\mathcal{G}\otimes tF[t])$ which has infinite codimension and which is maximal with respect to these properties. Then either $M = L(\mathcal{H},k,\alpha)\cap(\mathcal{G}\otimes tF[t])$, where $\mathcal{H}$ is as in 4.1(ii), or $M = L(\mathcal{G},pk,\beta)\cap(\mathcal{G}\otimes tF[t])$ where p,β are as in 4.1(iii). In any case, M is periodic.*

Again, the proof is omitted because of its close similarity to the proof of Theorems 1.1 and 1.2.

We conclude this paper with two explicit examples.

Example 4.3: Let F be a field of characteristic $\neq 2$ and let $\mathcal{G} = \mathrm{sl}_2(F)$. Put $L = \mathrm{sl}_2(tF[t]) \cong \mathcal{G}\otimes tF[t]$. Let $M = \bigoplus_{n\geq 1} M_n \otimes t^n$ be a graded subalgebra of L over F which is of infinite codimension and which is maximal with respect to these properties. Then one of the following holds:

(i) $M = \mathcal{H}\otimes tF[t]$ for some maximal subalgebra $\mathcal{H}$ of $\mathcal{G}$.

(ii) $M = \mathcal{G}\otimes t^pF[t^p]$ for some prime p.

(iii) There is a $\mathbb{Z}_2$-grading $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ (with $\dim\mathcal{G}_0 = 1$ and $\dim\mathcal{G}_1 = 2$) such that

$$M_n = \begin{cases} \mathcal{G}_0 & n \equiv 0 \bmod 2, \\ \mathcal{G}_1 & n \equiv 1 \bmod 2. \end{cases}$$

(iv) There is a $\mathbb{Z}$-grading $\mathcal{G} = \mathcal{G}_{-1}\oplus\mathcal{G}_0\oplus\mathcal{G}_1$ (with $\dim\mathcal{G}_i = 1$), an odd prime

p, and an integer $1 \le a \le (p-1)/2$, such that

$$M_n = \begin{cases} \mathcal{G}_0 & n \equiv 0 \bmod p, \\ \mathcal{G}_1 & n \equiv a \bmod p, \\ \mathcal{G}_{-1} & n \equiv -a \bmod p, \\ \{0\} & \text{otherwise.} \end{cases}$$

Moreover, up to conjugacy M can be described in a more explicit way. Suppose that $F = F^2$ (i.e. that F is closed under taking square roots), and let e, h, f denote the standard basis for $\mathcal{G}$. Then, up to conjugacy in $\operatorname{Aut}\mathcal{G}$, we have $\mathcal{H} = \langle e, h\rangle$ in (i), $\mathcal{G}_0 = \langle h\rangle$ and $\mathcal{G}_1 = \langle e, f\rangle$ in (iii), and $\mathcal{G}_1 = \langle e\rangle$, $\mathcal{G}_0 = \langle h\rangle$ and $\mathcal{G}_{-1} = \langle f\rangle$ in (iv).

Example 4.4: Let F be a field of characteristic $p \ge 5$, and let $\mathcal{G} = W_1$ be the first Witt algebra over F. Then $\mathcal{G}$ has a basis $e_0, \ldots, e_{p-1}$ satisfying $[e_i, e_j] = (j-i)e_{(i+j) \bmod p}$. Therefore $\mathcal{G}$ has a $\mathbb{Z}_p$-grading $\mathcal{G} = \bigoplus_{i=0}^{p-1} \mathcal{G}_i$ where $\mathcal{G}_i = \langle e_i\rangle$. Let α denote this grading. For $n \ge 1$, let $L_n = \mathcal{G}_{n \bmod p} \otimes t^n$, and let $L = \bigoplus_{n\ge 1} L_n = L(\mathcal{G}, p, \alpha) \cap (\mathcal{G} \otimes tF[t])$. We shall describe the graded subalgebras of infinite codimension in L which are maximal with respect to these properties. Let M be such an algebra. Then one of the following holds:

(i) There exists $1 \le a \le (p-1)/2$ such that

$$M = \bigoplus_{n \equiv 0, a, -a \bmod p} L_n.$$

(ii) There exists a prime $q \ne p$ such that

$$M = \bigoplus_{n \equiv 0 \bmod q} L_n.$$

Note that the subalgebras of type (ii) are isomorphic to the original Lie algebra L (by sending e_n to $q^{-1}e_{qn}$).

These descriptions can be easily derived using Theorem 4.2 and well known properties of sl_2 and W_1. We leave this as an exercise for the reader.

The Lie algebra in Example 4.3 occurs in the study of the first congruence subgroup of $\mathrm{SL}_2(F_p[[t]])$, while the Lie algebra in Example 4.4 occurs in the study of the so-called Nottingham group, namely the group of normalized automorphisms of $F_p[[t]]$. See [BSh] for more details.

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